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STACKS, COSTACKS AND AXIOMATIC HOMOLOGY

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Let $p: \mathscr{E} \to \mathfrak{X}$ be a sheaf (espace étalé) of abelian groups. Applying singular functor S one obtains a simplicial map $\pi: E \to X$ with $E = S(\mathscr{E})$, $X = S(\mathfrak{X})$ and $\pi = S(p)$. The fibers $\pi^{-1}(x)$, $x \in X$, form a "local system of groups" over X which will be called a costack of abelian groups over the simplicial set X. In general, a costack is defined as a functor on X, regarded as a category. This is a generalized dual of the notion of a stack defined by Spanier [5].

The main objects of this note are (1) to develop a general theory of stacks and costacks over simplicial sets, (2) to construct a semisimplicial homology theory with "variable" coefficients which is unique in the sense of Eilenberg-Steenrod. The coefficients of the homology are a costack in an abelian category. In particular, when the coefficient costack is a locally constant costack the homology becomes the usual homology with local coefficients.

There are three chapters in this note. Chapter I is devoted to a study of stacks and costacks. It is partially a preparatory chapter. In Chapter II we define homology of costacks via usual chain complexes and prove that the homology so defined can be computed by projective resolutions by introducing a generalized torsion product functor. Under the equivalence of costacks and modules, this generalized functor is essentially the genuine torsion product functor of modules. The rest of Chapter II is a preparation for Chapter III, in which a homology theory of pairs of simplicial sets over a fixed simplicial set K is defined. Results of Chapter II ensure the existence of such a theory. Chapter III concludes with a proof of the uniqueness of this homology theory. This is a generalization of Eilenberg-Steenrod uniqueness theorem [1].

The results presented in this note are a part of the author's Ph.D. thesis at the City University of New York written under the direction of Professor Alex Heller.

Chapter I. Stacks and Costacks

1. **Definitions and notations.** $X = \{X_q\}$ is a simplicial set (semisimplicial complex) regarded as a category with objects simplexes $x, x', \ldots$ and morphisms $\mu_x: x \to x'$ for incidence map μ such that $\mu(x) = x'$. The morphisms determined by face operators and degeneracy operators are denoted by d^i_x, s^i_x, or simply d, s. A simplicial

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map $f: X \to Y$ is thus a functor. If, as in [3], the simplicial set X is defined as a contravariant functor, then the associated category can be viewed as the *graph* of X.

Let $\mathscr{A}$ denote a category which has a projective generator P and satisfies the properties (1) $\mathscr{A}$ is abelian, (2) $\mathscr{A}$ is closed under the formation of products and coproducts (sums), and (3) the product and coproduct of a family of short exact sequences in $\mathscr{A}$ are short exact sequences in $\mathscr{A}$. E.g.: R-modules $\mathscr{M}_R$, abelian groups $\mathscr{A}b$, and compact abelian groups $\mathscr{A}b^*$ (the dual of $\mathscr{A}b$) are such categories. Since the category X is small (the class of objects is a set), the functor category $\mathscr{A}^X$ is well defined with morphisms natural maps of functors. $\mathscr{A}^X$ satisfies the three properties of $\mathscr{A}$ listed above.

An object $A \in \mathscr{A}^X$ is a functor $A: X \to \mathscr{A}$ which is called a *precostack* on X with values in $\mathscr{A}$). If A satisfies the condition that $A(s) = A(s^i_x)$ is an isomorphism for every s^i and x of X, then we say that A is a *costack*. Dually, *prestacks* and *stacks* are contravariant functors on X to $\mathscr{A}$.

2. **The functors $f_\#$ and $f^\#$.** A simplicial map $f: X \to Y$ induces functors $f_\#: A^X \to \mathscr{A}^Y$ and $f^\#: \mathscr{A}^Y \to \mathscr{A}^X$ as follows: For every A in $\mathscr{A}^X$, $f_\# A = B: Y \to \mathscr{A}$ is the functor defined on objects y and morphisms μ_y of Y as

$$(2.1) \qquad By = \coprod Ax, \qquad B\mu_y = \coprod A(\mu_x),$$

sum over all x such that $fx = y$ and over all μ_x such that $f(\mu_x) = \mu_y$. It is easy to check that $f_\#$ is a well defined functor. The functor $f^\#$ is defined by composition of functors as $f^\# B = Bf$ for B in $\mathscr{A}^Y$. Both $f_\#$ and $f^\#$ are exact functors and

PROPOSITION 2.1. *$f^\#$ is the adjoint of $f_\#$. i.e. there is a natural isomorphism*

$$(2.2) \qquad \mathscr{A}^X(A, f^\# B) \to \mathscr{A}^Y(f_\# A, B), \qquad A \in \mathscr{A}^X,\ B \in \mathscr{A}^Y.$$

Proof. Let $\varphi = \{\varphi_x \mid x \in X\}$ be in $\mathscr{A}^X(A, f^\# B)$, i.e. φ is a natural map with $\varphi_x: Ax \to (f^\# B)x$. Then, for $y \in Y$ and all $x \in X$ such that $fx = y$, the universal mapping diagram of $\coprod Ax$

$$\begin{array}{ccccc} Ax & \xrightarrow{i_x} & \coprod Ax & = & (f_\# A)_y \\ & \searrow^{\varphi_x} & \downarrow{\scriptstyle \psi_y} & & \downarrow \\ & & (f^\# B)x & = & By \end{array}$$

shows that the correspondence $\varphi \to \psi$ with $\varphi_x = \psi_y i_x$ defines a natural isomorphism.

Since $f_\#$ and $f^\#$ are exact, we have

COROLLARY 2.2. *$f_\#$ preserves projectives and $f^\#$ preserves injectives.*

For composite simplicial map gf we have $(gf)_\# = g_\# f_\#$ and $(gf)^\# = f^\# g^\#$.

3. **Projectives and generators in $\mathscr{A}^X$.** Let Δ^n denote the simplicial analogue of the unit affine n-simplex and let δ be its nondegenerate n-simplex. For every $x \in X_n$, the correspondence $\delta \to x$ determines uniquely a simplicial map $x^\delta \colon \Delta^n \to X$. We shall show that the induced functor $x^\delta_\# \colon \mathscr{A}^\Delta \to \mathscr{A}^X$ (here Δ stands for Δ^n) supplies projectives of $\mathscr{A}^X$.

THEOREM 3.1. *Let $P^\Delta \colon \Delta^n \to \mathscr{A}$ be the constant functor with value P (a projective generator of $\mathscr{A}$), then P^Δ is a projective of $\mathscr{A}^\Delta$.*

Proof. For any $F \in \mathscr{A}^\Delta$,

$$\mathscr{A}^\Delta(P^\Delta, F) \approx \mathscr{A}(P, F\delta). \tag{3.1}$$

For, let $\varphi = \{\varphi_\sigma \mid \sigma \in \Delta^n\}$ be in $\mathscr{A}^\Delta(P^\Delta, F)$, then the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi_\delta\ } & F\delta \\ {\scriptstyle \varphi_\sigma}\downarrow & & \downarrow{\scriptstyle F\sigma^*} \\ F\sigma & \xrightarrow{\ 1\ } & F(\sigma^*\delta), \end{array}$$

where σ^* is the incidence map of Δ^n determined by σ, shows that φ is completely determined by φ_δ and vice versa. Thus the correspondence $\varphi \to \varphi_\delta$ gives rise to the isomorphism 3.1. This and a routine computation show that P^Δ is projective.

THEOREM 3.2. *$U = \coprod_{x \in X} (x^\delta_\# P^\Delta)$ is a projective generator of $\mathscr{A}^X$.*

Proof. U is projective since $x^\delta_\#$ preserves projective, and coproduct of projectives is a projective. Now, a simple computation shows that

$$\mathscr{A}^X(U, A) \approx \prod \mathscr{A}^\Delta(P^\Delta, Ax^\delta) \approx \prod \mathscr{A}(P, Ax). \tag{3.2}$$

Thus $\mathscr{A}^X(U, A) \neq 0$ for any $A \neq 0$. U is a generator.

We conclude that since $\mathscr{A}^X$ has projective generators, it has enough projectives. Thus one can do homology in $\mathscr{A}^X$ by projective resolutions.

4. **Stacks and costacks.** A costack (resp. stack) as defined in §1 is a *normalized* precostack (resp. prestack). Since $A(d_{sx})A(s_x) = A(d_{sx}s_x) = 1$ for all $x \in X$, a precostack is normalized if and only if $A(d_{sx})$ is an isomorphism for all d_{sx}. The same holds true for stacks. In the rest of this paper, we shall leave out the dual theory for stacks.

Costacks form an abelian category $\bar{\mathscr{A}}^X$ which is an exact full subcategory of $\mathscr{A}^X$. It is easily shown that $\bar{\mathscr{A}}^X$ is a Serre subcategory of $\mathscr{A}^X$ in the sense that it is closed under the formation of subobjects, quotient objects and extensions. Also, $\bar{\mathscr{A}}^X$ is closed under the formation of products and coproducts. Thus, by a theorem of Freyd [2], we have

PROPOSITION 4.1. *$\bar{\mathscr{A}}^X$ is reflective and coreflective.*

$\bar{\mathscr{A}}^X$ is *coreflective* in the sense that for each $A \in \mathscr{A}^X$, there is $N^*A \in \bar{\mathscr{A}}^X$ and a map $r: A \to N^*A$ such that for any $\bar{A} \in \bar{\mathscr{A}}^X$ and any map $\varphi: A \to \bar{A}$ there is a unique map $\psi: N^*A \to \bar{A}$ with $\psi r = \varphi$. Reflectivity is defined dually.

The coreflector $N^*: \mathscr{A}^X \to \bar{\mathscr{A}}^X$ is the coadjoint of the inclusion functor $J: \bar{\mathscr{A}}^X \to \mathscr{A}^X$ and so preserves colimits. Since J is exact, N^* also preserves projectives. Thus

THEOREM 4.2. *Let NX be the set of nondegenerate simplexes of X, then $U^* = N^* \coprod_{x \in NX} (x^\delta_\# P^\Delta)$ is a projective generator of $\bar{\mathscr{A}}^X$.*

The reflection $\bar{A}$ of A is a costack defined as $\bar{A}x = Ax$ for $x \in NX$ and $A(sx) \approx Ax$ for all degeneracy operators s. The reflector N_* is exact and so its coadjoint functor J preserves projectives. Hence, a projective resolution of $\bar{A}$ in $\bar{\mathscr{A}}^X$ is also a projective resolution of $\bar{A}$ in $\mathscr{A}^X$. Summarizing, we say that $\bar{\mathscr{A}}^X$ is *homologically closed* in $\mathscr{A}^X$.

5. **Generalized torsion product functor.** For each $A \in \mathscr{A}^X$, let CA be the chain complex of objects in $\mathscr{A}$ with n-chains $\coprod_{x \in X_n} Ax$ and differential $\partial = \{\partial_n\}$ defined as

$$\partial_n = \coprod_{x \in X_n} \left(\sum_{i=0}^{n} (-1)^i A(d_x^i) \right). \tag{5.1}$$

The homology of CA is denoted by $H(A)$.

THEOREM 5.1. *On $\mathscr{A}^X$, H is naturally isomorphic to LH_0, the left derived functor of H_0.*

Proof. To show that for every projective A of $\mathscr{A}^X$, $H_n(A) = 0$ for $n > 0$. Since a projective is a summand of a coproduct of copies of projective generator U, it suffices to show that $H_n(U) = 0$ for $n > 0$. This is true since $CP^\Delta = C(x^\delta_\# P^\Delta)$ is acyclic and so is the coproduct $U = \coprod_{x \in X} (x^\delta_\# P^\Delta)$.

When X has finitely many nondegenerate simplexes then the category of costacks of abelian groups over X has a small projective generator U and may be identified with the category of right R modules, R is the endomorphism ring of U; H_q then becomes $\mathrm{Tor}^R_q(-, H_0 U)$.

EXAMPLE. If X is a simplicial complex, then $R \approx \coprod_{\sigma \leqq \tau} Z(\sigma, \tau)$, where $\sigma \leqq \tau$ means σ is a face of τ, $Z(\sigma, \tau)$ is the infinite cyclic group generated by the symbol (σ, τ). Observe that the multiplication in R is defined by

$$(\sigma, \rho)(\rho, \tau) = (\sigma, \tau); \qquad (\sigma, \rho)(\rho', \tau) = 0 \quad \text{if } \rho \neq \rho'. \tag{5.2}$$

CHAPTER II. HOMOLOGY WITH VARIABLE COEFFICIENTS

6. **Homology of simplicial pairs.** (X, X') is a simplicial pair with inclusion map $i: X' \to X$. The induced functor $i_\#: \mathscr{A}^{X'} \to \mathscr{A}^X$ maps $A': X' \to \mathscr{A}$ onto $i_\# A' = A: X \to \mathscr{A}$ with *supports* in X'. Precisely, $Ax = A'x$ for $x \in X'$ and $Ax = 0$ for $x \in X - X'$. $i_\#$ is an exact full embedding and $i^\# i_\#$ is the identity functor of $\mathscr{A}^{X'}$.

Observe that $i_{\#}i^{\#}A$ is a subobject of A with supports in X' and $i_{\#}i^{\#}$ is an exact reflector. If we identify $\mathscr{A}^{X'}$ with its image under $i_{\#}$, then

PROPOSITION 6.1. *$\mathscr{A}^{X'}$ is (identified as) a reflective Serre subcategory of $\mathscr{A}^{X}$.*

For every $A \in \mathscr{A}^X$, define qA by the exact sequence $0 \to i_{\#}i^{\#}A \to A \to qA \to 0$. qA has supports in $X - X'$. In fact, any object in $\mathscr{A}^X$ with supports in $X - X'$ is the quotient of some A by $i_{\#}i^{\#}A$. Such objects of $\mathscr{A}^X$ are called *relative precostacks*. They form a full subcategory $q\mathscr{A}^X$ of $\mathscr{A}^X$.

PROPOSITION 6.2. *$q\mathscr{A}^X$ is an exact coreflective Serre subcategory of $\mathscr{A}^X$. The coreflector q is exact.*

COROLLARY 6.3. *The functors $i_{\#}$ and q preserve projective resolutions.*

Similar statements are true for normalized categories $\bar{\mathscr{A}}^{X'}$, $\bar{\mathscr{A}}^{X}$ and $q\bar{\mathscr{A}}^{X}$.

Recall that for every $A \in \mathscr{A}^X$ there associates a chain complex CA, the homology of CA is denoted by $H(A)$. For a simplicial pair (X, X'), define its *homology with coefficients in* $A \in \mathscr{A}^X$ as $H(X, X'; A) = H(qA)$. In particular, $H(X; A) = H(A)$. Observe that if $f: (X, X') \to (Y, Y')$ is a simplicial map, then $C(f_{\#}A) = CA$ and $C(qf_{\#}A) = C(qA)$. Hence $H(X, X'; A) = H(Y, Y'; f_{\#}A)$. On the other hand, f induces a chain map $\{f_n\}: Cf^{\#}B \to CB$, $B \in \mathscr{A}^Y$, as follows: For n-chains

$$\coprod_x (f^{\#}B)x = \coprod_x Bf(x), \qquad x \in X_n,$$

and $\coprod_y By$, $y \in Y_n$, f_n is the unique map rendering the diagram

$$\begin{array}{ccc} Bf(x) & \xrightarrow{\;i_x\;} & \coprod_x Bf(x) \\ & \searrow^{i_y} & \downarrow f_n \\ & & \coprod_y By \end{array} \tag{6.1}$$

commutative. Thus f induces a map

$$f_*: H(X, X'; f^{\#}B) \to H(Y, Y'; B), \qquad B \in \mathscr{A}^Y. \tag{6.2}$$

PROPOSITION 7.4. *H is a functor in the sense that simplicial maps*

$$f: (X, X') \to (Y, Y') \quad \text{and} \quad g: (Y, Y') \to (K, K')$$

give rise to a map

$$(gf)_* = g_* f_*: H(X, X'; f^{\#}g^{\#}E) \to H(K, K'; E), \tag{6.3}$$

where $E \in \mathscr{A}^K$.

7. **Exactness, excision, additivity, and dimension.** From now on, all coefficients for homology are normalized. It is clear that the functor q, the functors $f^{\#}$ induced

by simplicial maps f, and the functor $i_{\#}$ induced by an inclusion map preserve normalization.

Let A be a coefficient costack, then the exact sequence $0 \to i_{\#}i^{\#}A \to A \to qA \to 0$ gives rise to

PROPOSITION 7.1 (EXACTNESS). *To each simplicial pair (X, X') is associated an exact homology sequence*

$$\cdots \to H_q(X'; i^{\#}A) \to H_q(X; A) \to H_q(X, X'; A) \to H_{q-1}(X'; i^{\#}A) \to \cdots,$$

where $i: X' \to X$ is the inclusion map. Moreover, if $f: (Y, Y') \to (X, X')$ is a simplicial map of pairs, then f induces a map f_* of homology sequences of the pairs.

Let $(X; X', X'')$ be a *triad* with inclusions

$$(X', X' \cap X'') \xrightarrow{i} (X' \cup X'', X'') \xrightarrow{h} (X, X' \cup X'')$$

$$(X'', X' \cap X'') \xrightarrow{j} (X' \cup X'', X'') \xrightarrow{h} (X, X' \cup X'').$$

It is easily shown that

PROPOSITION 7.2 (EXCISION). *The excision maps i and j induce isomorphisms*

$$i_*: H_*(X', X' \cap X''; i^{\#}h^{\#}A) \to H_*(X' \cup X'', X'; h^{\#}A)$$

$$j_*: H_*(X'', X' \cap X''; j^{\#}h^{\#}A) \to H_*(X' \cup X'', X''; h^{\#}A).$$

The following additivity properties of H are also easy to show.

PROPOSITION 7.3 (ADDITIVITY). *Given a simplicial pair (X, X') and a family $\{X_\alpha\}$ of simplicial subsets of X with the property that $X = X' \cup (\bigcup X_\alpha)$ and $X_\alpha \cap X_\beta \subset X'$ if $\alpha \neq \beta$. Let $X'_\alpha = X_\alpha \cap X'$ and let $h_\alpha: (X_\alpha, X'_\alpha) \to (X, X')$ be the inclusion map, then for any coefficient costack A we have*

$$H_*(X, X'; A) \approx \coprod_\alpha H_*(X_\alpha, X'_\alpha; h^{\#}_\alpha A).$$

In particular, when X' is void, we have

COROLLARY 7.4. *H is infinitely additive.*

Now, for each nondegenerate simplex x of X let Δ^x denote the simplicial subset of X determined by faces of x and let $\dot{\Delta}^x$ be its "boundary simplicial subset." If $i_x: \Delta^x \to X$ denotes the inclusion map then for any costack A on X the normalized chain complex of $q(i^{\#}_x A)$ has zero in all dimensions n except possibly for $n = \dim x$. Thus

PROPOSITION 7.5. $H_n(\Delta^x, \dot{\Delta}^x; i^{\#}_x A) = 0$ *for* $n \neq \dim x$.

8. **Strong homotopy and deformation.** For $n=0, 1, 2, \ldots$, let $I_n=[n, n+1]$, the closed unit interval as simplicial set, and let $W=\bigcup_{n=0}^{\infty} I_n$ be the "simplicial half line."

LEMMA 8.1. *For any constant costack E_X on X with value $E \in \mathscr{A}$, the projection $p\colon (X\times W, X'\times W) \to (X, X')$ defined by $p(x, \sigma)=x$ induces a chain equivalence*

$$C(p)\colon C(qp^{\#}E_X) \to C(E_X). \tag{8.1}$$

Proof. Let $\otimes\colon \mathscr{A}\times\mathscr{A}b \to \mathscr{A}$ be the tensor functor defined by Freyd [2, p. 86] and let $C(X, X'; Z)$ be the usual free chain complex of (X, X'). Then $C(E_X)\approx E\otimes C(X, X'; Z)$ and $C(qp^{\#}E_X)\approx E\otimes C(X\times W, X'\times W; Z)$. It is well known that p induces a chain equivalence of the free chain complexes. This gives rise to the chain equivalence (8.1).

LEMMA 8.2. *Let NX_n denote the set of all nondegenerate n-simplexes of X. Then for any coefficients*

$$H_*(X^n, X^{n-1}) \approx \coprod_{x\in NX_n} H_*(\Delta^x, \dot{\Delta}^x). \tag{8.2}$$

This follows immediately from Proposition 7.3.

PROPOSITION 8.3 (STRONG HOMOTOPY). *$p\colon X\times W \to X$ induces isomorphism*

$$p_*\colon H_X(X\times W; p^{\#}A) \to H_*(X; A) \tag{8.3}$$

for any coefficient costack A.

Proof. First, we shall show by induction that

$$H_*(X^n\times W; p^{\#}A) \approx H_*(X^n; A) \tag{8.4}$$

for any nonnegative integer n. The crucial point is the fact that $(p^{\#}A)(x, \sigma)=Ap(x, \sigma)=Ax$ for all $\sigma\in W$ and then $H_*(\Delta^x, \dot{\Delta}^x)$ and $H_*(\Delta^x\times W, \dot{\Delta}^x\times W)$ *have constant coefficients* for any fixed $x\in NX$.

For the case $n=0$, $H_*(X^0\times W)=\coprod_{x\in X_0} H_*(\Delta^x\times W)$ is isomorphic to

$$\coprod_{x\in X_0} H_*(\Delta^x)$$

since, by Lemma 8.1, each summand $H_*(\Delta^x\times W)$ is isomorphic to $H_*(\Delta^x)$. Hence we have $H_*(X^0\times W)\approx H_*(X^0)$.

Assume inductively that $H_*(X^r\times W)\approx H_*(X^r)$ for $r=1, 2, \ldots, n-1$, and consider the commutative diagram

$$\begin{array}{ccccccccc}
\cdots\to & H_q(X^{n-1}\times W) & \to & H_q(X^n\times W) & \to & H_q(X^n\times W, X^{n-1}\times W) & \to & H_{q-1}(X^{n-1}\times W) & \to\cdots \\
 & \downarrow 2 & & \downarrow 3 & & \downarrow 4 & & \downarrow 5 & \\
\cdots\longrightarrow & H_q(X^{n-1}) & \longrightarrow & H_q(X^n) & \longrightarrow & H_q(X^n, X^{n-1}) & \longrightarrow & H_{q-1}(X^{n-1}) & \longrightarrow\cdots,
\end{array}$$

where the maps 2 and 5 are isomorphisms. Since

$$H_*(X^n \times W, X^{n-1} \times W) \approx \coprod_{x \in NX_n} H_*(\Delta^x \times W, \dot{\Delta}^x \times W)$$

and $H_*(X^n, X^{n-1}) \approx \coprod_{x \in X_n} H_*(\Delta^x, \dot{\Delta}^x)$ by Lemma 8.2, it follows from Lemma 8.1 that the map 4 is an isomorphism. Hence, by the five lemma, the map 3 is an isomorphism. This proves (8.4) and, of course, the case when X is finite dimensional.

Now, suppose that X is infinite dimensional with $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$. Clearly, $H_q(X^n) = H_q(X^{n-1}) = \cdots = H_q(X)$ for $n > q+1$. This and (8.4) prove (8.3).

COROLLARY 8.4 (HOMOTOPY). *Let p: $X \times I \to X$ be the simplicial map defined by $p(x, \sigma) = x$ for $x \in X$ and any $\sigma \in I$, then for any $A \in \mathscr{A}^X$, p_*: $H_*(X \times I; p^\# A) \to H_*(X; A)$ is an isomorphism.*

For, we have retractions $X \times W \xrightarrow{r'} X \times I \xrightarrow{r} X \times [0]$ such that $r_* r'_* = (rr')_*$ is an isomorphism.

PROPOSITION 8.5 (DEFORMATION). *The projection p: $\bigcup_n X^n \times I_n \to X$ defined by $p(x, \sigma) = x$, where $(x, \sigma) \in X^n \times I_n$, $n = 0, 1, 2, \ldots$, induces isomorphism*

$$(8.5) \qquad p_*: H_*(L; p^\# A) \to H_*(X; A), \qquad L = \bigcup_n X^n \times I_n.$$

Proof. Let $L^n = \bigcup_{r=0}^n X^r \times I_r$ and let $LX^n = L^n \cup (X^n \times [n+1, \infty))$, then $L^n \subset LX^n \subset L$. Since LX^n is a deformation retract of $X^n \times W$, $H_q(LX^n) \approx H_q(X^n \times W)$. Hence, by Proposition 8.3, $H_q(LX^n) \approx H_q(X^n) \approx H_q(X)$ for $n > q+1$. Thus for any $q \geqq 0$, there is $n > q+1$ such that

$$H_q(X) \approx H_q(LX^n) \approx H_q(LX^{n+1}) \approx \cdots \approx H_q(L).$$

The proof is complete.

CHAPTER III. HOMOLOGY THEORY ON $\mathscr{C}'_K$

9. ***K*-pairs and axioms for homology.** Let K be a fixed simplicial set. A *K-pair* is a simplicial pair (X, X') together with a simplicial map φ: $X \to K$. Such a K-pair is denoted by $(X, X')_\varphi$. $(K, K')_1$ is written (K, K') and $(X, \phi)_\varphi$ is written X_φ. When $\varphi = \sigma^\delta$: $\Delta^q \to K$, the subscript σ^δ is abbreviated by σ.

Given two K-pairs $(X, X')_\varphi$ and $(Y, Y')_\psi$, a *K-map* f: $(X, X')_\varphi \to (Y, Y')_\psi$ is, by definition, a simplicial map f: $(X, X') \to (Y, Y')$ such that $\varphi = \psi f$. In particular, an inclusion map i: $(Y, Y') \to (X, X')$ is a *K-map* i: $(Y, Y')_{\varphi i} \to (X, X')_\varphi$ for any simplicial map φ: $X \to K$. $(Y, Y')_{\varphi i}$ is called a *K-subpair* of $(X, X')_\varphi$. We shall omit the inclusion map in the notation of a K-subpair. E.g.: write $(Y, Y')_\varphi$ for $(Y, Y')_{\varphi i}$, X'_φ for $X'_{\varphi i}$, X_φ for $X_{\varphi i}$, $(\Delta^x, \dot{\Delta}^x)_\varphi$ for $(\Delta^x, \dot{\Delta}^x)_{\varphi i}$, etc.

K-pairs form a category, denoted by $\mathscr{C}_K$, with morphisms K-maps. Any K-pair of the form (K, K') is a terminal object (right zero object).

A homology theory on $\mathscr{C}'_K$ with values in the category $\mathscr{A}$ is a sequence of functors $H_*: \mathscr{C}'_K \to \mathscr{A}$ together with a family of natural transformations $\partial_q: H_q(X, X')_\varphi \to H_{q-1}X_\varphi$, $q>0$, satisfying the following axioms:

Axiom 1 (*Exactness axiom*). For each $(X, X')_\varphi$ with inclusion maps $X'_\varphi \xrightarrow{i} X_\varphi \xrightarrow{j} (X, X')_\varphi$ there is an *exact triangle* of $(X, X')_\varphi$,

$$\text{(9.1)} \qquad \begin{array}{ccc} H_*H'_\varphi & \xrightarrow{i_*} & H_*H_\varphi \\ & \searrow^{\partial} & \downarrow j_* \\ & & H_*(X, X')_\varphi, \end{array}$$

where $i_* = H_*i$, $j_* = H_*j$.

Let $j_0, j_1: (X, X') \to (X \times I, X' \times I)$ and $p: (X \times I, X' \times I) \to (X, X')$ be simplicial maps defined by $j_0x = (x, 0)$, $j_1x = (x, 1)$, and $p(x, \sigma) = x$, respectively, where $x \in X$, $\sigma \in I$. Then for any simplicial map $\varphi: X \to K$, j_0, j_1, and p are K-maps as shown in the commutative diagram

$$\text{(9.2)} \qquad \begin{array}{ccccc} X & \xrightarrow{j_\alpha} & X \times I & \xrightarrow{p} & X \\ & \searrow^{\varphi} & \downarrow \varphi p & \swarrow^{\varphi} & \\ & & K, & & \alpha = 0, 1. \end{array}$$

Two K-maps $f, g: (X, X')_\varphi \to (Y, Y')_\psi$ are *K-homotopic* if there is a K-map $h: (X \times I, X' \times I)_{\varphi p} \to (Y, Y')_\psi$, called a *K-homotopy* of f and g, such that $f = hj_0$, $g = hj_1$.

Axiom 2 (*Homotopy axiom*). p_* induced by the K-projection $p: (X \times I, X' \times I)_{\varphi p} \to (X, X')_\varphi$ is an isomorphism, or equivalently, if f and g are K-homotopic then $f_* = g_*$.

Axiom 3 (*Excision axiom*). The excision maps i and j of §7 regarded as K-maps induce isomorphisms i_* and j_*.

For the dimension axiom we need the following argument: In analogy to ordinary simplicial homology theory, let C^{q-1} be the closed star of a vertex in $\dot{\Delta}^q$ [1, p. 78], then $(\Delta^q; \Delta^{q-1}, C^{q-1})_\sigma$ is a proper triad with respect to H_*. This and the exactness axiom give rise to the diagram

$$\text{(9.3)} \qquad \begin{array}{ccccc} H_q(\Delta^q, \dot{\Delta}^q)_\sigma & \xrightarrow{\partial} & H_{q-1}(\dot{\Delta}^q)_\sigma. & \xrightarrow{h_*} & H_{q-1}(\dot{\Delta}^q, C^{q-1})_\sigma \\ & \searrow^{F^i} & & & \downarrow j_*^{-1} \\ & & & & H_{q-1}(\Delta^{q-1}, \dot{\Delta}^{q-1})_{d\sigma}, \end{array}$$

where $d\sigma=\tau$ is the ith face of $\sigma \in k$, h is an inclusion map, j is an excision map, and $F^i=j_*^{-1}h_*\partial$.

Axiom 4 (*Dimension axiom*). For any $x \in NX_q$ with $\varphi x=\sigma$, x_*^δ: $H_q(\Delta^q, \dot{\Delta}^q)_\sigma \to H_q(\Delta^x, \dot{\Delta}^x)$ is an isomorphism and $H_n(\Delta^q, \dot{\Delta}^q)_\sigma=0$ for $n \neq q$. If $\sigma=s^i\tau$, then F^i defined by (9.3) is an isomorphism.

Axiom 5 (*Additivity axiom*). Let $(X_\alpha, X'_\alpha)_\varphi$ be K-subpairs of $(X, X')_\varphi$ defined as in Proposition 7.3, then

$$H_*(X, X')_\varphi \approx \coprod_\alpha H_*(X_\alpha, X'_\alpha)_\varphi.$$

Axiom 6 (*Deformation axiom*). $p_*=H_*(p)$, where p is the K-map $p: L_{\varphi p} \to X_\varphi$ defined as in Proposition 8.5, is an isomorphism.

REMARK. These axioms are of course modelled on those of Eilenberg-Steenrod [1] supplemented by Milnor's additivity axiom [4]. If $\mathscr{A}$ satisfies AB5 (exactness of directed colimits) they could be somewhat abbreviated by supposing that directed colimits were preserved. We must avoid this supposition if we are to have a selfdual theory: it is false even for group-valued cohomology, i.e. homology with values in $\mathscr{A}b^*$.

10. **Existence theorem, coefficient costacks.** Let A be a costack on K with values in $\mathscr{A}$. For each $(X, X')_\varphi \in \mathscr{C}'_K$, let

$$H_*((X, X')_\varphi; A) = H_*(X, X'; \varphi^\# A), \tag{10.1}$$

the right-hand side is the homology of the simplicial pair (X, X') with coefficients in $\varphi^\# A$ as defined in the previous chapter.

If $f: (X, X')_\varphi \to (Y, Y')_\psi$ is a K-map, then $\psi f=\varphi$ and so $f^\#\psi^\#=\varphi^\#$. We then have $H_*((X, X')_\varphi; A)=H_*(X, X'; f^\#\psi^\# A)$. The map $f_*: H_*(X, X'; f^\#\psi^\# A) \to H_*(Y, Y'; \psi^\# A)$ is the induced map $H_*f: H_*((X, X')_\varphi; A) \to H_*((Y, Y')_\psi; A)$. The results of Chapter II show that

THEOREM 10.1 (EXISTENCE THEOREM). *For every costack A on K there is a homology theory H_* on $\mathscr{C}'_K$ defined by the chain homology functor as*

$$H_*((X, X')_\varphi; A) = H_*(q\varphi^\# A).$$

Now, let H_* be any homology theory on $\mathscr{C}'_K$. The *coefficient costack* A of H_* is, by definition, the costack on K with $A\sigma=H_q(\Delta^q, \dot{\Delta}^q)_\sigma$ for $\sigma \in K$ and with $A(d^i)=F^i$, $A(s^i)=(F^i)^{-1}$. We observe that the coefficient costack of the homology theory H_* in the theorem is just that A.

If K is a point, a K-pair is just a pair of simplicial sets and the theory H_* in the theorem is the ordinary simplicial homology with local coefficients.

11. **Uniqueness theorem.** We shall show that the H_* in Theorem 10.1 is essentially the only homology theory on $\mathscr{C}'_K$.

THEOREM 11.1 (UNIQUENESS THEOREM). *Let h_* be any homology theory on $\mathscr{C}'_K$. There is a natural isomorphism*

$$h_*(X, X')_\varphi \approx H_*((X, X')_\varphi; A), \tag{11.1}$$

where A is the coefficient costack of the theory h_*.

Proof. First we show (11.1) for finite dimensional case. Let $\phi = X^{-1} \subset X^1 \subset X^2 \subset \cdots \subset X^r = X_\varphi$ (the subscripts φ in X^n_φ are omitted) be the increasing filtration of X_φ by skeletons. It is an easy consequence of the dimension axiom that the associated spectral sequence collapses and that $h_*(X_\varphi)$ is naturally isomorphic to the homology of the chain complex C^h with

$$C^h_q = H_q(X^q, X^{q-1}) \approx \coprod_x H_q(\Delta^x, \dot{\Delta}^x), \qquad x \in NX_q.$$

It follows from the dimension axiom and the definition of A that

$$C^h_q \approx \coprod_x H_q(\Delta^q, \dot{\Delta}^q)_{\varphi x} = \coprod_x A\varphi(x), \qquad x \in NX_q.$$

Thus $C^h_q \approx \coprod_x (\varphi^\# A)x = C_q(\varphi^\# A)$. From the constructions of A and C^h we observe that $C^h \approx C(\varphi^\# A)$ as chain complexes. Hence $h_*(X_\varphi) \approx H_*(X_\varphi; A)$. Therefore (11.1) follows from the exactness axiom and the five lemma.

Next, suppose that X is infinite dimensional. We have seen that it suffices to prove the isomorphism for the absolute case. From the first part of this proof, we see that for a fixed integer $q \geqq 0$ and any integer $n > q$ there is a canonical isomorphism $h_q(X^n) \approx H_q(X^n; A)$. But

$$H_q(X^n_\varphi; A) \approx H_q(X^{n+1}; A) \approx \cdots \approx H_q(X_\varphi; A), \tag{11.2}$$

we have a direct system

$$h_q(X^0) \xrightarrow{i^0_*} h_q(X^1) \xrightarrow{i^1_*} h_q(X^2) \xrightarrow{i^2_*} \cdots \tag{11.3}$$

with isomorphisms i^n_* for $n > q+1$.

Now, use Axioms 1, 2, 3, 5, and 6 and proceed as in [4], we get a Mayer-Vietoris sequence

$$\begin{array}{ccc} \coprod_{n=0}^{\infty} h_*(X^n) & \xrightarrow{\ f\ } & \coprod_{n=0}^{\infty} h_*(X^n_\varphi) \\ & {\nwarrow}\,\partial' \qquad q\,{\swarrow} & \\ & h_*(X_\varphi) & \end{array}$$

with $\operatorname{Coker} f_q = \lim h_q(X^n)$. Dual to the Lemma 2 of [4], denote the kernel of f_q by $\mathscr{L}'\{h_{q-1}(X^n)\}$ and call $\mathscr{L}'$ the derived functor of lim, then there is an exact sequence

$$0 \to \lim h_q(X^n) \to h_q(X) \to \mathscr{L}'\{h_{q-1}(X^n)\} \to 0$$

and a similar one for H_*. Apply (11.2) and (11.3), we have $\mathscr{L}'\{h_{q-1}(X^n)\}=0$ and $h_q(X_\varphi)\approx H_q(X_\varphi; A)$.

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